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ON SYNTHESIS IN A DIFFERENTIAL GAME*

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The control problem is considered with minimization of the guaranteed result for a system described by an ordinary differential equation in the presence of uncontrolled noise. The concepts and formulation of the problem in /1/ are used. It is shown that, when forming the optimal control by the method of programmed stochastic synthesis /1-3/, the extremal shift at the accompanying point /1, 4/ can be reduced to extremal shift agianst the gradient of the appropriate function. This explains the connection between the programmed stochastic synthesis and the generalized Hamilton-Jacobi equation /5, 6/ in the theory of differential games.

1. Formulation of the problem. Consider the system described by the differential equation

$$x^{\cdot} = A(t)x + f(t, u, v),\ u \in P,\ v \in Q,\ t_0 \leqslant t \leqslant \vartheta \tag{1.1}$$

Here, x is the n-dimensional phase vector of the object, u is the r-dimensional control vector, v is the s-dimensional noise vector, $A(t)$ is a continuous matrix function, $f(t, u, v)$ is a continuous vector function, P and Q are compacta, and

$$\gamma = \int\limits_{[t_*, \vartheta]} \sigma(t, x[t])\,\mu(dt) + \int\limits_{t_*}^{\vartheta} \chi(t, u[t], v[t])\,dt \tag{1.2}$$

The functional which characteristizes the quality of the process in an interval $[t_*, \vartheta] \subset [t_0, \vartheta]$ is given. Here, $\sigma(t, x)$ and $\chi(t, u, v)$ are scalar continuous functions, $\sigma(t, x)$ satisfies a Lipschitz condition and is convex with respect to x, and $\mu(T)$ is the Borel measure in sets $T \subset [t_0, \vartheta]$.

We consider motions $x[t_*[\cdot]\vartheta] = \{x[t], t_* \leqslant t \leqslant \vartheta\}$, lying in a given bounded domain G of space $\{t, x\}$. Domain G is defined for $t_0 \leqslant t \leqslant \vartheta$, is closed, and satisfies the following condition /1, pp.37-42/. Given any initial position $\{t_*, x_*\} \in G$, every possible motion $x[t_*[\cdot]\vartheta]$ satisfies the inclusion $\{t, x[t]\} \in G$ for all $t \in [t_*, \vartheta]$. The problem is to construct the optimal strategy $u^\circ(\cdot) = \{u^\circ(t, x, \varepsilon), \{t, x\} \in G, \varepsilon > 0\}$, which gives the minimum guaranteed result $\rho^\circ(t_*, x_*)$.

This strategy exists and by definition, satisfies the following condition /1, pp.67-81/. Given any number $\zeta > 0$, a number $\varepsilon(\zeta) > 0$ and a function $\delta(\zeta, \varepsilon) > 0$ exist such that the control law

$$U = \{u^\circ(\cdot), \varepsilon, \Delta\{t_i\}\} \tag{1.3}$$

which forms the motion as a solution of the step-by-step differential equation

$$\begin{gathered} x^{\cdot}[t] = A(t)x[t] + f(t, u^\circ(t_i, x[t_i], \varepsilon), v[t]) \\ t_i \leqslant t < t_{i+1},\ i = 1, \ldots, k,\ t_1 = t_*,\ t_{k+1} = \vartheta,\ x[t_*] = x_* \end{gathered} \tag{1.4}$$

guarantees the inequality $\gamma \leqslant \rho^\circ(t_*, x_*) + \zeta$, no matter what the measurable noise

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$v[t_*[\cdot]\vartheta) = \{v[t] \in Q,\ t_* \leqslant t < \vartheta\}$, provided that $\varepsilon \leqslant \varepsilon(\zeta)$, $\delta \leqslant \delta(\zeta, \varepsilon)$. Then, $\rho = \rho^\circ(t_*, x_*)$ is the least of the numbers ρ which satisfy the similar condition.

2. Programmed stochastic extremum. We can assume without loss of generality that the Lipschitz constant λ with respect to x for the function $\sigma(t, x)$ and the measure $\mu(T)$ in (1.2) satisfy in the domain G the conditions /1, p.380/ $\lambda \leqslant 1$, $\mu([t_0, \vartheta]) \leqslant 1$. We can arrange for this by changing the scale of measuring γ and without thereby distorting the problem. It can be shown /1, Chapter V/ that we have for $\rho^\circ(t_*, x_*)$ the equation

$$\rho^\circ(t_*, x_*) = \sup \beta,\ \beta \in B(t_*, x_*) \tag{2.1}$$

for the set $B(t_*, x_*)$ of numbers β given by

$$B(t_*; x_*) = \{\beta\colon \sup_\Delta e(t_*, \{x_*, 0\}, \Delta, \beta) > \beta\} \tag{2.2}$$

Here, an $(n+1)$-dimensional vector of the type $\{x, z_{n+1}\} = z = \{z_1, \ldots, z_n, z_{n+1}\}$, makes an appearance. The first n components of z form the vector $x = \{x_1, \ldots, x_n\} = \{z_1, \ldots, z_n\}$. In (2.2), $x = x_*$, $z_{n+1} = 0$. By Δ we denote the division $\Delta\{\tau_j\}$ of the interval $[t_*, \vartheta]$ by points τ_j, $j = 1, \ldots, k$, $\tau_1 = t_*$, $\tau_{k+1} = \vartheta$, k is an integer.

What we called in /1/ the programmed extremum $e(t_*, z_*, \Delta, \beta)$ is given by

$$e(t_*, z_*, \Delta, \beta) = \sup_{\|l(\cdot)\| \leqslant 1} \{\varkappa(t_*, z_*, \Delta\{\tau_j\}, \beta, l(\cdot))\} + \beta \tag{2.3}$$

Here, $l(\cdot) = \{l(\tau, \omega) = l[\tau, \xi_1, \ldots, \xi_k],\ t_* \leqslant \tau \leqslant \vartheta,\ \omega = \{\xi_1, \ldots, \xi_k\} \in \Omega\}$ is an n-dimensional random vector function. It is defined in the auxiliary probability space $\{\Omega, F, \mathbf{P}\}$; ξ_j $(j = 1, \ldots, k)$ are independent in aggregate random quantities, each of which is uniformly distributed in the half-interval $0 \leqslant \xi_j < 1$. The quantity $\|l(\cdot)\|$ is any suitable norm for $l(\cdot)$ in the direct product of spaces $\{[t_0, \vartheta], B^T, \mu\} \otimes \{\Omega, F, \mathbf{P}\}$.

For example, to be specific we take

$$\|l(\cdot)\| = \Big(\int\limits_{[t_*, \vartheta]} \int\limits_{\Omega} |l[\tau, \xi_1, \ldots, \xi_k]|^2\, d\xi_1 \ldots d\xi_k\, d\tau\Big)^{1/2} \tag{2.4}$$

where $|l|$ is the Euclidean norm of the vector l. In (2.3), $\varkappa$ is given by

$$\begin{aligned}\varkappa(t_*, z_*, \Delta\{\tau_j\}, \beta, l(\cdot)) = \langle s_* \cdot x_*\rangle + z_{*n+1} + \\ \int\limits_{t_*}^{\vartheta} M\{\min_{u \in P} \max_{v \in Q} [\langle s(\tau, \omega) \cdot f(\tau, u, v)\rangle + \chi(\tau, u, v)]\}\, d\tau - \\ \sup_{r(\cdot) \in R_\beta} M\Big\{\int\limits_{[t_*, \vartheta]} \langle l(\tau, \omega) \cdot w(\tau, \omega)\rangle\, \mu(d\tau) + r_{n+1}(\omega)\Big\}\end{aligned} \tag{2.5}$$

Here, $\langle\cdot\rangle$ denotes the scalar product of vectors, and $M\{\ldots\}$ is the expectation in space $\{\Omega, F, \mathbf{P}\}$. The vector s_* is given by

$$s_* = \int\limits_{[t_*, \vartheta]} M\{X'[\eta, t_*]\, l(\eta, \omega)\}\, \mu(d\eta) \tag{2.6}$$

where $X[t, \tau]$ is the fundamental matrix of solutions of the differential equation $dx/dt = A(t)x$; the prime denotes transposition. The random vector function $s(\tau, \omega)$ is given by

$$\begin{aligned} s(\tau, \omega) = M\Big\{\int\limits_{[\tau, \vartheta]} X'[\eta, \tau]\, l(\eta, \omega)\, \mu(d\tau)\,|\, \xi_1, \ldots, \xi_j\Big\} \\ \tau_j \leqslant \tau < \tau_{j+1},\quad j = 1, \ldots, k \end{aligned} \tag{2.7}$$

where $M\{\ldots|\,\xi_1, \ldots, \xi_j\}$ is the conditional expectation with respect to $\xi_1, \ldots, \xi_j$. By R_β we denote the set of $(n+1)$-dimensional random vector functions

$$\begin{gathered} r(\cdot) = \{r(\tau, \omega) = \{r_1(\tau, \omega), \ldots, r_n(\tau, \omega);\quad r_{n+1}(\omega)\} = \\ \{w(\tau, \omega); r_{n+1}(\omega)\},\ t_* \leqslant \tau \leqslant \vartheta,\ \omega \in \Omega\} \end{gathered}$$

constrained by the condition

$$\int\limits_{[t_*, \vartheta]} \sigma(\tau, w(\tau, \omega))\, \mu(d\tau) + r_{n+1}(\omega) \leqslant \beta$$

for almost all $\omega \in \Omega$.

3. Approximation of the optimal guaranteed result. We introduce the vector function

$$m[\tau] = M\{l(\tau, \omega)\}, \quad t_* \leqslant \tau \leqslant \vartheta \tag{3.1}$$

We put

$$l(\tau, \omega) = m[\tau] + b(\tau, \omega) \tag{3.2}$$

On substituting (3.2) for $l(\tau, \omega)$ into (2.3)-(2.7) and using (3.1), we obtain

$$\sup_\Delta e(t_*, z_*, \Delta, \beta) = \max_{\|m[\cdot]\| \leqslant 1} (\langle s_*(t_*, m[\cdot]) x_* \cdot \rangle + z_{*n+1} + \varkappa^*(t_*, m[\cdot], \beta)) \tag{3.3}$$

Here, by (2.4), the norm $\|m[\cdot]\|$ is given by

$$\|m[\cdot]\| = \Big(\int_{[t_*, \vartheta]} |m[\tau]|^2 \mu(d\tau)\Big)^{1/2} \tag{3.4}$$

The functional $\langle s_*(t_*, m[\cdot]) \cdot x_*\rangle$ is linear with respect to $m[\cdot]$. The functional $\varkappa^*(t_*, m[\cdot], \beta)$ is concave in $m[\cdot]$. This important fact, which follows from the stochastic nature of the function $l(\cdot)$, is proved by arguments similar to those in /1, pp.311-314/.

Let

$$\lambda^* = \max_{t_* \leqslant \tau \leqslant \vartheta} \max_{|x|=1} |A(\tau) x|$$

We consider in space $\{z\}$ the domain

$$K(t_*, x_*, \varepsilon) = \{z\colon |w - x_*|^2 + z_{n+1}^2 \leqslant \alpha^2(t_*, \varepsilon)\}$$
$$(\alpha(\tau, \varepsilon) = (\varepsilon + \varepsilon \exp 2\lambda^*(\tau - t_0))^{1/2})$$

We construct the quantity

$$\rho(t_*, x_*, \varepsilon, \beta) = \min_{z \in K} \sup_\Delta e(t_*, z, \Delta, \beta) = \min_{z \in K} \max_{\|m[\cdot]\| \leqslant 1} (\langle s_*(t_*, m[\cdot]) \cdot w\rangle + z_{n+1} + \varkappa^*(t_*, m[\cdot], \beta)), \; z = \{w, z_{n+1}\} \tag{3.5}$$

Since $s_*(t_*, m[\cdot])$ is linear, and $\varkappa^*(t_*, m[\cdot], \beta)$ is concave, with respect to $m[\cdot]$, and $\langle s_*(t_*, m[\cdot]) \cdot w\rangle$ is linear with respect to w, the operations of min with respect to $z = \{w, z_{n+1}\}$ and of max with respect to $m[\cdot]$ in (3.5) can be interchanged. We thus obtain the equation

$$\rho(t_*, x_*, \varepsilon, \beta) = \max_{\|m[\cdot]\| \leqslant 1} (\langle s_*(t_*, m[\cdot]) \cdot x_*\rangle - \alpha(t_*, \varepsilon)[1 + |s_*(t_*, m[\cdot])|^2]^{1/2} + \varkappa^*(t_*, m[\cdot], \beta)) \tag{3.6}$$

The maximized quantity is concave with respect to $m[\cdot]$ and strictly concave with respect to $s_*(t_*, m[\cdot])$. For fixed t_*, x_*, ε, and β, therefore, the vector $s^\circ(t_*, x_*, \varepsilon, \beta) = s_*(t_*, m^\circ[\cdot])$, corresponding to a maximizing function $m^\circ[\cdot]$, is uniquely defined. We define $\beta(t_*, x_*, \varepsilon)$ as the upper limit

$$\beta(t_*, x_*, \varepsilon) = \sup \beta, \quad \beta \in B(t_*, x_*, \varepsilon) \tag{3.7}$$

where the set $B(t_*, x_*, \varepsilon)$ is given by the condition

$$B(t_*, x_*, \varepsilon) = \{\beta\colon \rho(t_*, x_*, \varepsilon, \beta) > \beta\} \tag{3.8}$$

We can seek $\beta(t_*, x_*, \varepsilon)$ as the least root of the equation $\rho(t_*, x_*, \varepsilon, \beta) = \beta$.

Consider the function

$$\rho^*(t_*, w, x_*, \varepsilon) = \rho(t_*, w, \varepsilon, \beta(t_*, x_*, \varepsilon)) \tag{3.9}$$

With fixed t_*, x_*, and ε it has the gradient $\operatorname{grad}_w \rho^*(t_*, w, x_*, \varepsilon)$, which is continuous with respect to w and satisfies the equation

$$\operatorname{grad}_w \rho^*(t_*, w, x_*, \varepsilon) = s^\circ(t_*, w, \varepsilon, \beta(t_*, x_*, \varepsilon)) \tag{3.10}$$

These assertions, may be proved by starting from (3.6)-(3.9) and using the uniqueness of the vector $s^\circ(t_*, w, \varepsilon, \beta(t_*, x_*, \varepsilon))$.

Denote by $z^\circ(t_*, x_*, \varepsilon)$ the accompanying point /1, p.209/, at which $\rho^\circ(t_*, w) + z_{n+1}$ reaches its minimum in the domain $K(t_*, x_*, \varepsilon)$. From (2.1) and (2.2) and the above working, it may be seen that the $(n+1)$-dimensional vector

$$p^\circ(t_*, x_*, \varepsilon) = \{x_*, 0\} - z^\circ(t_*, x_*, \varepsilon) \tag{3.11}$$

which connects the points $z^\circ(t_*, x_*, \varepsilon)$ and $\{x_*, 0\}$, is connected with the vector $s^\circ(t_*, x_*, \varepsilon, \beta(t_*, x_*, \varepsilon))$ by the equation

$$p^\circ(t_*, x_*, \varepsilon) = \alpha(t_*, \varepsilon)\{s^\circ(t_*, x_*, \varepsilon, \beta(t_*, x_*, \varepsilon)), 1\} \times (1 + |s^\circ(t_*, x_*, \varepsilon, \beta(t_*, x_*, \varepsilon))|^2)^{-1/2} \tag{3.12}$$

The optimal control $u^\circ(t_i, x[t_i], \varepsilon)$ can be formed according to the law U of (1.3), which /1, p.231/ denotes extremal shift to the accompanying point $z^\circ(t_i, x[t_i], \varepsilon)$. In view of (3.12), this amounts to extremal shift in opposition to the vector

$$\{s^\circ(t_i, x[t_i], \varepsilon, \beta(t_i, x[t_i], \varepsilon)), 1\} = \{[\mathrm{grad}_w \rho^*(t_i, w, x[t_i], \varepsilon)]_{w=x[t_i]}, 1\}$$

We thus arrive at the following conclusion.

The optimal min-max strategy, which gives the minimum guaranteed result $\rho^\circ(t_*, x_*)$ for any initial position $\{t_*, x_*\} \in G$, can be constructed as the function $u^\circ(t, x, \varepsilon)$, which satisfies the condition of extremal shift opposite to the vector $\{[\mathrm{grad}_w \rho^*(t, w, x, \varepsilon)]_{w=x}, 1\}$, i.e., from the condition

$$\max_{v\in Q} \{\langle [\mathrm{grad}_w \rho^*(t, w, x, \varepsilon)]_{w=x} \cdot f(t, u^\circ(t, x, \varepsilon), v)\rangle + \chi(t, u^\circ(t, x, \varepsilon), v)\} = \min_{u\in P} \max_{v\in Q} \{\langle [\mathrm{grad}_w \rho^*(t, w, x, \varepsilon)]_{w=x} \cdot f(t, u, v)\rangle + \chi(t, u, v)\} \tag{3.13}$$

4. Notes. In many specific problems it is possible to find the maximizing division $\Delta\{\tau_j^\circ\}$, by which the upper bound with respect to Δ is reached in (3.3). The construction of the function $\rho^*(t, w, x, \varepsilon)$ or of its required gradient $[\mathrm{grad}_w \rho^*(t, w, x, \varepsilon)]_{w=x}$ is then simplified. If the division $\Delta\{\tau_j^\circ\}$, on which the upper bound is reached in (3.3), is not discovered, then, when constructing in practice the optimal control signals $u^\circ[t_i] = u^\circ(t_i, x[t_i], \varepsilon)$, there is no need to take evaluation of the function $\rho^*(t, w, x, \varepsilon)$ as far as evaluation of this upper bound with respect to Δ. We can confine ourselves to choosing a division $\Delta\{\tau_j^*\}$ with a sufficiently small step $\tau_{j+1}^* - \tau_j^* \leqslant \delta$, because it can be shown that every sequence of divisions $\Delta\{\tau_j^{(k)}\}$ $(k = 1, 2, \ldots)$ with step $\delta \overset{(k)}{\to} 0$ as $k \to \infty$ is maximizing for problem (3.3). Also, the upper limit with respect to β in (3.7), (3.8) may be computed approximately.

The above procedure for evaluating the control signal $u^\circ[t_i]$ can often be greatly simplified by selecting a suitable norm $\|l(\cdot)\|$ for the function $l(\cdot)$ and accordingly, a suitable norm for the function $m[\cdot]$. For instance, if

$$\mathrm{vraimax}_\omega \left[\int_{[t_*, \vartheta]} \sigma(\tau, w(\tau, \omega))\, \mu(d\tau) + |r_{n+1}(\omega)| \right]$$

signifies the norm in space R of random vector functions $r(\tau, \omega) = \{w(\tau, \omega), r_{n+1}(\omega)\}$, the constraint on $l(\cdot)$ in the maximization problem with respect to $l(\cdot)$ is determined by the norm, conjugate to the norm in R. This can prove convenient also, because in this case the quantity β can disappear from the working altogether. Then, the function $\rho^*(t, w, x, \varepsilon)$ is likewise simplified, since its third argument x can disappear. In general, elimination of β from the working can be conveniently achieved in many cases when the function $\sigma(t, x)$ is homogeneous with respect to x. The constraint on $l(\cdot)$ in the maximization problem can then be conveniently chosen as a constraint on a suitable conjugate functional of $l(\cdot)$.

This procedure is also convenient in cases when $\chi(t, u, v) \equiv 0$. The introduction of the supplementary coordinate z_{n+1} regularizes the problem. After permuting the min and max operations in (3.5), (3.6), the resolving vector $s^\circ(t_*, w, x_*, \varepsilon, \beta)$ on the right-hand side of (3.10) is unique. This determines the differentiability of $\rho^*(t_*, w, x_*, \varepsilon)$ of (3.9) with respect to w and gives Eqs.(3.10).

It must be emphasized that the definition of the optimal max-min counter-strategy $v^\circ(t, x, u, \varepsilon)$ from the condition of extremal shift along the vector $\{[\mathrm{grad}_w \rho^*(t, w, x, \varepsilon)]_{w=x}, 1\}$ is not in general well-posed, if the cost $\rho^\circ(t, x)$ of the differential game is not differentiable with respect to x. In such cases the optimal counter-actions $v^\circ[t] = v^\circ(t_i, x[t_i], u[t], \varepsilon)$ can be constructed on the basis of the optimal max-min unanticipated stochastic programs, which are extracted from the solution of the problem on evaluating the programmed extremum $e(\cdot)$ /1, p.420/.

Finally, notice the connection of the method of stochastic synthesis with the construction of convex hulls for the functions or functionals figuring in Sect.3. This connection enables us to organize procedures for computing $\varkappa^*$, based on a recurrence construction of these convex hulls of functionals. In some cases such procedures work effectively. In general, however, they cannot be considered effective, since the construction of the convex hulls of functionals or functions present a difficult problem in practice.

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ON THE RELATIVISTIC THEORY OF ROCKET FLIGHT*

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It is shown by macroscopic analysis that, when the entire mass of a rocket is consumed for creating thrust, objects may be obtained as a result having energy but zero mass, moving the velocity of light. It is shown that the boost process of such massless objects can be realized in finite time from the observer's point of view. The vast stellar luminosity of quasars and certain jet motions observed in remote space can be explained by the production of massless radiation with internal motions connected with the separation of large energies inside the stars.

A number of publications have dealt with rocket motions in the context of relativistic effects. One of the first was by Ackeret /1/, then there was Sanger's /2/, while other authors largely took these as a basis for their first principles. It should be mentioned that some authors have sometimes used super-light relative velocities of the rejected masses, which is not admissible.

Sanger gave a detailed theory of inhabited relativistic rockets with equipages in board, allowing for the arrival at the rocket of opposed cosmic masses, used as energy sources in reactive motors of the stright-through type.

Below we study limit relations for the motions of uninhabited rockets, when their consumed rest mass tends to zero.

A typical feature of rocket motion is connected with the rejection of mass belonging to them and the consequent creation of thrust, so that start and boost of the rocket can be realized with a considerable increase in their flight velocity relative to a fixed observer.

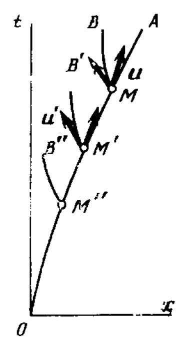

Fig.1

In Fig.1 we show schematically the world line of the rocket in the Cartesian inertial system of observer's reading xt. An arbitrary rocket position is denoted by M; $M''B''$, $M'B'$, and MB are elements of the world lines for the rejected infinitesimal masses $|\Delta m^k|$ by means of the rocket motors, and $\mathbf{u}_k'$ is their initial four-dimensional velocity (the gas flow velocity from the motor nozzles). In general it is natural to assume that the further motion of the rejected masses along world lines MB cannot affect the motion of rocket M, though the properties of the initial vector $\mathbf{u}_k'$ are extremely important.

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